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Bipartite consensus of integrator multi-agent systems with measurement noise

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Abstract: The bipartite consensus problem for integrator multi-agent systems over signed fixed digraphs is investigated in the presence of measurement noise. A time-varying consensus gain is introduced and then a stochastic type protocol is proposed, whose performance is analysed using the state transition matrix of the closed-loop system. Necessary and sufficient conditions for ensuring a mean square bipartite consensus protocol are obtained in the presence of noise. Furthermore, in the absence of noise it is shown that these conditions are also necessary and sufficient for ensuring the bipartite consensus except for the quadratic integrability of the consensus gain. It is found that the signed digraph being structurally balanced and having a spanning tree are the weakest assumptions on connectivity for achieving bipartite consensus regardless of the measurement noise. In particular, if the signed digraph is structurally unbalanced, then under some mild conditions, the states of the closed-loop system converge to zero in mean square, regardless of the initial states.

1 Introduction

The research on consensus control of multi-agent systems (MASs) has been active over the past two decades [1–7]. In MASs the agents may compete or cooperate with each other, and consensus is reached by their cooperative interactions. In particular, consensus may be such that the agents have the same value but with different signs, referred to as bipartite consensus [8]. Note that bipartite consensus is different from the well-known group consensus [3], in that the latter may have multiple consensus groups and each group may have different, independent consensus values. Examples of bipartite consensus can be found in diverse fields including sociology [9], physics [10] and so on. It is known that bipartite consensus usually relies on three key factors, i.e. the agent dynamic structure, the consensus protocols and the communication topology among agents. Existing studies often keep one or two such factors fixed and then investigate the role of the rest factors. For example, under given agent dynamic structure and bipartite consensus protocol, the role of the communication topology has been studied in [8, 11–14] and so on. In the pioneering work [8], it is shown that for first-order integrator MASs with linear Laplacian-like bipartite consensus protocols and strongly connected signed digraph, bipartite consensus is achieved if and only if the signed digraph is structurally balanced. Then, the connectivity condition in [8] is relaxed in [11] to having a spanning tree. The fixed communication topology in [8] is extended in [12] to time-varying signed digraphs. Moreover, bipartite consensus for high-order MASs with unknown disturbances is investigated [15–17]. In [15], an adaptive consensus protocol is designed for second-order MASs. In [16, 17], for high-order MASs, a leader, whose information is only available to a part of agents, is introduced to intervene a group of agents and a neural network-based adaptive protocol is proposed. Since a Lyapunov function candidate is usually hard to be constructed for high-order MASs, a common Lyapunov function method is employed in the convergence analysis [16, 17].

On the other hand, measurement noise is usually inevitable in practice and has been studied widely in the context of conventional consensus control [18–24]. To name a few, a stochastic approximation-type protocol with a decreasing step size is proposed for discrete-time models in [18]; the continuous-time integrator models are considered in [19], where necessary and sufficient conditions for ensuring mean square average consensus are given. Subsequently, the protocols developed in [19] are extended in [20] to time-varying topologies, in [21] to double-integrator MASs and in [22] to MASs described by a continuous-time linear time-invariant model with single input. Moreover, multiplicative noises are considered in [23, 24] and so on. In [23], the noise intensity is assumed to be proportional to the absolute value of the relative states of an agent and its neighbours, and sufficient conditions to achieve mean square consensus and strong consensus are given by properly selecting constant consensus gains, respectively. In [24], the intensity function in [23] is further extended to a vector function.

However, measurement noise has not been well studied for bipartite consensus to date, thus motivating our this study. Two reasons make conventional stochastic analysis tools [18–20] unsuitable for bipartite consensus, justifying the necessity of our work. Firstly, in bipartite consensus, cooperation and competition coexist and interactions among agents are generally represented by signed graphs with positive/negative weights, which are different from conventional consensus. This means that the row sum of *Laplacian* may not be zero and *Laplacian* of a signed graph can be positive definite, different from standard graph theory for conventional consensus. Secondly, the error in conventional consensus is merely the difference, while for bipartite consensus it might be the sum of agent state and the final consensus value, making the state matrix of the error dynamics and *Laplacian* coupled, and therefore causing another challenge.

In this work, we design distributed protocols to reach bipartite consensus in the presence of measurement noise. Inspired by [18, 19], a time-varying consensus gain is introduced, leading to a time-varying stochastic system as the closed-loop system. It is worth pointing out that in [25], stochastic Lyapunov function is constructed to analyse the closed-loop system and only sufficient conditions are given to ensure mean square weak bipartite consensus under undirected signed graphs. Here, directed signed graphs are considered. By using the theory of state transition matrix and Cauchy criterion, the dynamic characteristics of the closed-loop system are fully described. In the case with noise, necessary and sufficient conditions are obtained to describe the proposed protocol under which a mean square bipartite consensus is achieved. It is proven that the states of all the agents converge to a common random vector except for the sign. The mathematical expectation of this random vector is determined by the initial states

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and communication topology among agents. In the case without noise, some of the above conditions are also necessary and sufficient for ensuring a bipartite consensus. It turns out that the conditions for the signed digraph to be structurally balanced and having a spanning tree are not only sufficient but also necessary for achieving bipartite consensus. In particular, if the signed digraph is structurally unbalanced, the states of the closed-loop system will converge to zero in mean square under some mild conditions, regardless of the initial states.

The remaining of the paper is organised as follows. In Section 2, we first review some basic concepts and properties associated with signed graphs, and then formulate the problem of interest, with the introduction of some necessary lemmas. The main results are then presented in Section 3. A numerical example is given in Section 4, and Section 5 concludes the paper.

Notations: The following notations are used throughout this paper. $\mathrm{Re}(\lambda)$ is the real part of λ. $\mathbf{1}_N$ is an N-dimensional column vector with all ones. $\mathbf{0}$ is the zero matrix with appropriate dimension. $\mathrm{tr}(A)$ and $\mathrm{rank}(A)$ are the trace and rank of A, respectively. For a given matrix or vector X, X^{T} is the transpose of X, $\| X \|$ its Euclidean norm, $\| X \|_{\mathrm{F}}$ its Frobenius norm, $\| X \|_1$ the 1-norm and $\| X \|_\infty$ the ∞ norm, respectively. For a random vector x, $E(x)$ is its mathematical expectation and $D(x)$ its variance. $\otimes$ is the Kronecker product.

2 Preliminaries and problem formulation

2.1 Preliminaries on signed graph

A signed digraph can be represented by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \ldots, N\}$ is the set of nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. An edge of $\mathcal{E}$ is denoted by an ordered pair (j, i), where j is the parent vertex of i, and i is the child vertex of j. $N_i = \{ j \in \mathcal{V} \mid (j, i) \in \mathcal{E} \}$ denotes the neighbour set of agent i. $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix. $a_{ij} \neq 0 \Leftrightarrow (j, i) \in \mathcal{E}$. $a_{ij} > 0$ means cooperation and $a_{ij} < 0$ means competition between agent i and j. Throughout this paper, we always assume that $a_{ii} = 0, i \in \mathcal{V}$ and $a_{ij} a_{ji} \geq 0$, $\forall i, j \in \mathcal{V}$. $\mathcal{L} = \mathcal{C}_r - \mathcal{A}$ is *Laplacian* of $\mathcal{G}$, where $\mathcal{C}_r = \mathrm{diag}(\sum_{j=1}^{N} |a_{1j}|, \ldots, \sum_{j=1}^{N} |a_{Nj}|)$. Let $\mathcal{C}_c = \mathrm{diag}(\sum_{j=1}^{N} |a_{j1}|, \ldots, \sum_{j=1}^{N} |a_{jN}|)$, and a signed digraph is weight balanced if $\mathcal{C}_r = \mathcal{C}_c$.

A digraph is a directed tree if each of its vertices except the root has exactly one parent. A spanning tree of a digraph is a directed tree that contains all the vertices of the digraph.

A graph $\hat{\mathcal{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}}, \hat{\mathcal{A}})$ is a subgraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ if $\hat{\mathcal{V}} \subseteq \mathcal{V}$, $\hat{\mathcal{E}} \subseteq \mathcal{E}$ and all non-zero elements of $\hat{\mathcal{A}}$ belong to $\mathcal{A}$. $\hat{\mathcal{G}}$ is an input-isolated subgraph if no edge comes from $\mathcal{V} \setminus \hat{\mathcal{V}}$ to $\hat{\mathcal{V}}$, i.e. $a_{ji} = 0, \forall j \in \hat{\mathcal{V}}, i \in \mathcal{V} \setminus \hat{\mathcal{V}}$.

A signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced if there exist two subsets $\mathcal{V}_1, \mathcal{V}_2$, $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$ such that $a_{ij} \geq 0, \forall i, j \in \mathcal{V}_k (k \in \{1, 2\})$, $a_{ij} \leq 0, \forall i \in \mathcal{V}_l, j \in \mathcal{V}_k$, $k \neq l (k, l \in \{1, 2\})$. It is structurally unbalanced otherwise.

Laplacian of a signed digraph is closely related to the concept of structural balance, as stated below.

Lemma 1 [13, 26]: Laplacian $\mathcal{L}$ of a signed digraph $\mathcal{G}$ has at least one zero eigenvalue and all of the non-zero eigenvalues are in the open right half-plane, if $\mathcal{G}$ is structurally balanced. Furthermore, $\mathcal{L}$ has exactly one zero eigenvalue if and only if $\mathcal{G}$ has a spanning tree.

Lemma 2: Let $\mathcal{G}$ be a signed digraph with *Laplacian* $\mathcal{L}$. Then, all the eigenvalues of $\mathcal{L}$ are in the open right half-plane if and only if $\mathcal{G}$ is neither structurally balanced nor contains an input-isolated structurally balanced (ISB) subgraph.

Proof: Consider system $\dot{X}(t) = -\mathcal{L}X(t)$. It is clear that $\forall X(0)$, $\lim_{t \to \infty} X(t) = \mathbf{0}$ is equivalent to that all the eigenvalues of $\mathcal{L}$ are in the open right half-plane.

According to Theorem 1 in [12], $\lim_{t \to \infty} X(t) = \mathbf{0}$ is equivalent to the statement that $\mathcal{G}$ is neither structurally balanced nor contains an ISB subgraph. This completes the proof. □

2.2 Problem formulation

Suppose that the agents have the following dynamics:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, \ldots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ are the state and control input for the ith agent, respectively.

Assume that the ith agent receives information from its neighbours with measurement noise $z_{ji}(t) = x_j(t) + \varphi_{ji}(t), j \in N_i$, $i = 1, \ldots, N$, where $\varphi_{ji}(t), j \in N_i, i = 1, \ldots, N$ is n-dimensional independent standard white noise.

The main objective here is to design a distributed protocol under measurement noise in order that for any initial value the states of agents in (1) converge to values with equal modulus (they may have different signs though). To that end, a time-varying consensus gain $a(t)$ is introduced for each agent, as follows:

$$u_i(t) = a(t) \left[\sum_{j \in N_i} |a_{ij}| \big(\mathrm{sgn}(a_{ij}) z_{ji}(t) - x_i(t) \big) \right], \tag{2}$$

$i = 1, \ldots, N$, where $a(t) > 0$ is a piecewise continuous function. We know that for conventional consensus a similar protocol to (2) may be designed by dropping $\mathrm{sgn}(a_{ij})$, which guarantees that each agent changes its state such that it gets close to its parent. Here, due to the existence of competition in our problem setting and the use of the term $\mathrm{sgn}(a_{ij})$, the protocol in (2) guarantees that each agent changes its state in such a way that it gets close to its cooperative neighbours but far away from its compete neighbours, and finally reaching bipartite consensus.

Let $X(t) = \big(x_1^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t) \big)^{\mathrm{T}}$. Given the protocol in (2), the state of the integrator agents in (1) evolves as follows:

$$\mathrm{d}X(t) = -a(t)(\mathcal{L} \otimes I_n) X(t) \, \mathrm{d}t + a(t)(B \otimes I_n) \, \mathrm{d}W(t), \tag{3}$$

where $B = \mathrm{diag}(\eta_1^{\mathrm{T}}, \ldots, \eta_N^{\mathrm{T}})$ is an $N \times N^2$ block diagonal matrix with $\eta_i^{\mathrm{T}} = (a_{i1}, \ldots, a_{iN})$, $W(t) = \big(W_1^{\mathrm{T}}(t), \ldots, W_N^{\mathrm{T}}(t) \big)^{\mathrm{T}}$, $W_i(t) = (W_{i1}^{\mathrm{T}}(t), \ldots, W_{Ni}^{\mathrm{T}}(t))^{\mathrm{T}}, i = 1, \ldots, N$, $\int_0^t \varphi_{ji}(s) \, \mathrm{d}s = W_{ji}(t)$ and $W_{ji}(t), i, j = 1, \ldots, N$ are standard Brownian motions.

The closed-loop system in (3) is a time-varying stochastic differential equation. To characterise the asymptotic behaviour of agents in (3), we first introduce the definition of bipartite consensus for stochastic system.

Definition 1: A distributed protocol $\mathcal{U} = \{u_i, i = 1, \ldots, N\}$ is a mean square bipartite consensus protocol if for any given $X(0) \in \mathbb{R}^{nN}$, there exist $\rho = (\rho_1, \ldots, \rho_N)^{\mathrm{T}} \in \mathbb{R}^N (\rho_i \in \{ \pm 1 \}$, $i = 1, \ldots, N)$ and random vector c^* such that

$$\lim_{t \to \infty} E \| X(t) - \rho \otimes c^* \|^2 = 0,$$

where $E \| c^* \|^2 < \infty$, Ec^* is determined by the initial state $X(0)$ and the communication topology among agents.

Definition 1 shows that in the non-trivial case ($c^* \neq 0$), the states of the agents converge to c^* or $-c^*$ in mean square.

Definition 2: A distributed protocol $\mathcal{U} = \{u_i, i = 1, \ldots, N\}$ is said to be mean square stabilising if it renders system (1) to satisfy $\lim_{t \to \infty} E \| X(t) \|^2 = 0, \forall X(0) \in \mathbb{R}^{nN}$.

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Finally, we list the following assumptions which may be used in the lemmas and theorems that follows:

(H_1) $\int_0^\infty a(s)\,\mathrm{d}s = \infty$.

(H_2) $\int_0^\infty a^2(s)\,\mathrm{d}s < \infty$.

(H_3) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced.

(H_4) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ has a spanning tree.

Remark 1:

(i) (H_1) and (H_2) are often used in stochastic approximation theory [18]. A possible choice of $a(t)$ satisfying both (H_1) and (H_2) was given in [19]: $C_1[\log t]^{\kappa_1}/t^{\alpha_1} \le a(t) \le C_2[\log t]^{\kappa_2}/t^{\alpha_2}$, where $\kappa_1 \le 1$, $\kappa_2 > -0.5$, $\alpha_1 \le 1$, $\alpha_2 > 0.5$, $C_1, C_2 > 0$.
(ii) From Theorem 1 in [12] it is known that (H_3) and (H_4) are the weakest assumptions on communication topology for ensuring a bipartite consensus in the absence of noise.

Remark 2:

(i) In the absence of noise, i.e. $\varphi_{ji}(t) \equiv 0$, let $a(t) \equiv 1$, the protocol in (2) is exactly the same bipartite consensus protocol as in [8, 11–13, 26]. However, in the presence of noise, for constant $a(t)$ the resulting evolution of agent states leads to fluctuations. To deal with measurement noise, the time-varying consensus gain $a(t)$ is thus introduced.
(ii) If all communication weights $a_{ij} \ge 0$, $i, j = 1, \ldots, N$ in $\mathcal{G}$, then $\mathcal{G}$ becomes a standard digraph in conventional consensus. Under this situation, (2) is the same protocol as in [19, 20]. Furthermore, only the states of agent i and its neighbours are used and hence (2) is a distributed protocol.

Remark 3:

(i) In Definition 1, the requirement on Ec^* is necessary, since otherwise agents in (1) will not have to communicate with each other, regardless of the initial value $X(0)$ nor the communication content among agents. In such a case one needs only to design protocols based on states of themselves (for example, taking $u_i(t) = -x_i(t)$, then one has $x_i(t) \to 0, t \to \infty$).
(ii) Definition 1 here is not equivalent to that in [25]. The latter requires the convergence of the state differences/sums between/of different agents, regardless of the convergence of the states themselves.

In the absence of measurement noise, the states of the agents are all deterministic processes. Then Definition 1 is equivalent to bipartite consensus in [8, 12, 13, 26].

2.3 Useful lemmas

To derive the main results, some useful lemmas are presented first.

Lemma 3: The state transition matrix of the following system is $\Phi_{J_r^\lambda}(t, t_0) \triangleq e^{-\int_{t_0}^t a(s)\,\mathrm{d}s J_r^\lambda} \otimes I_n$,

$$\frac{\mathrm{d}Z_r}{\mathrm{d}t} = -a(t)(J_r^\lambda \otimes I_n)Z_r, \quad r \in \mathbb{N}, \lambda \in \mathbb{C}, t \ge t_0 \ge 0, \tag{4}$$

where $Z_r = (x_{r1}^{\mathrm{T}}, x_{r2}^{\mathrm{T}}, \ldots, x_{rN_r}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{nN_r}$, and J_r^λ is a $N_r \times N_r$ Jordan block with λ on its diagonal. In addition, $\lim_{t\to\infty} \Phi_{J_r^\lambda}(t, t_0) = \mathbf{0}$ if $\int_0^\infty a(s)\,\mathrm{d}s = \infty$ and $\mathrm{Re}(\lambda) > 0$.

Proof: By definition

$$e^{-\int_{t_0}^t a(s)\,\mathrm{d}s J_r^\lambda} \triangleq \sum_{k=0}^{\infty} \frac{(-\int_{t_0}^t a(s)\,\mathrm{d}s)^k (J_r^\lambda)^k}{k!}.$$

By direct calculation, one has $e^{-\int_{t_0}^t a(s)\,\mathrm{d}s J_r^\lambda} = e^{-\lambda\int_{t_0}^t a(s)\,\mathrm{d}s}$

$$\times \begin{pmatrix} 1 & -\int_{t_0}^t a(s)\,\mathrm{d}s & \cdots & \frac{(-\int_{t_0}^t a(s)\,\mathrm{d}s)^{N_r-1}}{(N_r-1)!} \\ & 1 & \ddots & \vdots \\ & & \ddots & -\int_{t_0}^t a(s)\,\mathrm{d}s \\ & & & 1 \end{pmatrix}. \tag{5}$$

Therefore, $\Phi_{J_r^\lambda}(t_0, t_0) = e^{-\int_{t_0}^{t_0} a(s)\,\mathrm{d}s J_r^\lambda} \otimes I_n = I_{nN_r}$ and $\dot{\Phi}_{J_r^\lambda}(t, t_0) = -a(t)(J_r^\lambda \otimes I_n)\Phi_{J_r^\lambda}(t, t_0)$, which implies that $\Phi_{J_r^\lambda}(t, t_0)$ is the state transition matrix of (4).
In addition, if $\int_0^\infty a(s)\,\mathrm{d}s = \infty$ and $\mathrm{Re}(\lambda) > 0$, then

$$\lim_{t\to\infty} \left| \frac{(-\int_{t_0}^t a(s)\,\mathrm{d}s)^m}{m!} e^{-\lambda\int_{t_0}^t a(s)\,\mathrm{d}s} \right| = \lim_{t\to\infty} \frac{(\int_{t_0}^t a(s)\,\mathrm{d}s)^m}{m! e^{\mathrm{Re}(\lambda)\int_{t_0}^t a(s)\,\mathrm{d}s}} = 0, \quad m = 0,$$

$1, \ldots, N_r - 1$.

The above together with (5) leads to $\lim_{t\to\infty} \Phi_{J_r^\lambda}(t, t_0) = \mathbf{0}$. □

Let $\Phi(t, t_0)(t_0 \ge 0)$ be the state transition matrix of the closed-loop system in (3). Clearly, by the definition of matrix exponential function, $\Phi(t, t_0) = e^{-\int_{t_0}^t a(s)\,\mathrm{d}s \mathscr{L}} \otimes I_n$. If *Laplacian* $\mathscr{L}$ has exactly one zero eigenvalue and all of the non-zero eigenvalues are in the open right half-plane, then there exists an invertible matrix T such that

$$T^{-1}\mathscr{L}T = J = \mathrm{diag}(0, J_2, \ldots, J_s), \tag{6}$$

where $J_i (i = 2, \ldots, s)$ is an $N_i \times N_i$ Jordan block with λ_i on its diagonal, $N_2 + \cdots + N_s = N - 1$. Obviously, $\lambda_2, \ldots, \lambda_s$ are eigenvalues of $\mathscr{L}$ and $\mathrm{Re}(\lambda_i) > 0$, $i = 2, \ldots, s$. This together with Lemma 3 gives the state transition matrix $\Phi(t, t_0) = (T \otimes I_n)[\mathrm{diag}(I_n, \Phi_{J_2^{\lambda_2}}(t, t_0), \ldots, \Phi_{J_s^{\lambda_s}}(t, t_0))](T^{-1} \otimes I_n)$,
where $\Phi_{J_k^{\lambda_k}}, k = 2, \ldots, s$ are defined as in Lemma 3. In addition, if $\int_0^\infty a(s)\,\mathrm{d}s = \infty$, then

$$\lim_{t\to\infty} \Phi(t, t_0) = \left[T\mathrm{diag}(1, 0, \ldots, 0)T^{-1}\right] \otimes I_n. \tag{7}$$

Thus there exists $M > 0$ such that

$$\max\left(\| \Phi(t, t_0) \|_1, \| \Phi(t, t_0) \|_\infty \right) \le M < \infty, \quad \forall t \ge t_0 \ge 0. \tag{8}$$

If the protocol in (2) is a mean square bipartite consensus protocol, the following lemma will provide a useful insight into the state transition matrix $\Phi(t, t_0)(t_0 \ge 0)$ of the closed-loop system in (3). It plays an essential role in the asymptotic behaviour analysis.

Lemma 4: If the protocol in (2) is a mean square bipartite consensus protocol, then there exist $\rho(\rho_1, \ldots, \rho_N \in \{\pm 1\})$ and $\upsilon = (\upsilon_1, \ldots, \upsilon_N)^{\mathrm{T}} \in \mathbb{R}^N$ such that $\lim_{t\to\infty} \Phi(t, 0) = \rho\upsilon^{\mathrm{T}} \otimes I_n$.
If the protocol in (2) is mean square stabilising, i.e. $\lim_{t\to\infty} E \| X(t) \|^2 = 0, \forall X(0)$, then $\lim_{t\to\infty} \Phi(t, 0) = \mathbf{0}$.

Proof: If the protocol in (2) is a mean square bipartite consensus protocol, then according to Definition 1, for any given $X(0)$, there exist ρ and random vector c^* such that $\lim_{t\to\infty} E \| X(t) - (\rho \otimes c^*) \|^2 = 0$. From (3), one immediately obtains $X(t) = \Phi(t, 0)X(0) + \int_0^t a(s)\Phi(t, s)(B \otimes I_n)\,\mathrm{d}W(s)$. Without

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loss of generality, one may assume that $\int_0^t a(s)\Phi(t,s)(B\otimes I_n)\,\mathrm{d}W(s)$ converges to Y^* in mean square. Thus, $\rho\otimes Ec^* = \lim_{t\to\infty} EX(t) = \lim_{t\to\infty}\Phi(t,0)X(0) + EY^* \triangleq (\Phi_\infty \otimes I_n)X(0) + EY^*$, where $\lim_{t\to\infty}\Phi(t,0)\triangleq\Phi_\infty\otimes I_n$. Noting that $X(0)$ is arbitrary. Hence, for $2X(0)$ and $3X(0)$, there exist $\tilde{\rho}\in\mathbb{R}^N$ ($\tilde{\rho}_i = \pm 1$), $\tilde{c}^*$ and $\hat{\rho}\in\mathbb{R}^N(\hat{\rho}_i = \pm 1)$, $\hat{c}^*$ such that $2(\Phi_\infty\otimes I_n)X(0) + EY^* = \tilde{\rho}\otimes E\tilde{c}^*$ and $3(\Phi_\infty\otimes I_n)X(0) + EY^* = \hat{\rho}\otimes E\hat{c}^*$. Therefore, one derives that

$$\rho\otimes Ec^* = \tilde{\rho}\otimes 2E\tilde{c}^* - \hat{\rho}\otimes E\hat{c}^*. \tag{9}$$

Then we must have $\rho = \pm\tilde{\rho} = \pm\hat{\rho}$. If otherwise, take $\rho\neq\pm\tilde{\rho} = \pm\hat{\rho}$ as an example. In this case, besides $Ec^* = 2E\tilde{c}^* - E\hat{c}^*$, (9) contains at least one n-dimensional component which is $Ec^* = -2E\tilde{c}^* + E\hat{c}^*$. So $Ec^* = \mathbf{0}$. This contradicts the statement that Ec^* is determined by $X(0)$ and the communication topology among agents. Other cases can be similarly eliminated.

Combining $\rho = \pm\tilde{\rho} = \pm\hat{\rho}$ with the above equations one has

$$(\Phi_\infty\otimes I_n)X(0) = \rho\otimes Ed^*, \tag{10}$$

where Ed^* is a linear combination of Ec^*, $E\tilde{c}^*$ and $E\hat{c}^*$. Let $\Phi_\infty = (\xi_1,\ldots,\xi_N)$. Then each column ξ_j has entries with equal absolute value. Analogously, any linear combination $\sum_{j=1}^N(\xi_j\otimes\alpha_j)(\alpha_j\in\mathbb{R}^n, j=1,\ldots,N)$ has n-dimensional components with equal modulus.

If $\Phi_\infty\equiv\mathbf{0}$, then by taking $\upsilon = \mathbf{0}$, the statement of Lemma 4 is true. If at least one column of Φ_∞, say, ξ_1 is non-zero, then $\xi_1 = \upsilon_1\rho$ ($\upsilon_1\neq 0$, say, $\upsilon_1 > 0$). Note that for $\forall\alpha,\beta\in\mathbb{R}^n$, $a,b\in\mathbb{R}$, $a,b\neq 0$, we have $a\alpha + b\beta\neq\pm(a\alpha - b\beta)$. Therefore, if $\xi_j\neq\mathbf{0}$ for some $j\neq 1$, all n-dimensional components of $\xi_j\otimes\alpha - \xi_1\otimes\beta$ have equal modulus if and only of $\xi_j = \upsilon_j\rho$, $\upsilon_j > 0$. If $\xi_j = \mathbf{0}$, let $\upsilon_j = 0$, and then $\xi_j = \upsilon_j\rho$ for any j and $\Phi_\infty = (\xi_1,\ldots,\xi_N) = (\upsilon_1\rho,\ldots,\upsilon_N\rho) = \rho(\upsilon_1,\ldots,\upsilon_N)$. Noticing that $\lim_{t\to\infty}\Phi(t,0)\triangleq\Phi_\infty\otimes I_n$, we have $\lim_{t\to\infty}\Phi(t,0) = \rho(\upsilon_1,\ldots,\upsilon_N)\otimes I_n$.

If the protocol in (2) is mean square stabilising, then repeating the above arguments, we have $\lim_{t\to\infty}\Phi(t,0) = \mathbf{0}$. □

The following lemma is instrumental for the main result in the following section.

Lemma 5: If assumptions $(H_1)-(H_4)$ hold, then for any given $X(0)$, there exists random vector X^* such that $X(t)$ of the closed-loop system in (3) converges to X^* in mean square, i.e. $\lim_{t\to\infty} E\parallel X(t) - X^*\parallel^2 = 0$.

Proof: Since (H_3) and (H_4) hold, Lemma 1 implies that $\mathscr{L}$ has exactly one zero eigenvalue and all of the non-zero eigenvalues are in the open right half-plane. This together with assumption (H_1) gives (7) and (8). By *Itô* formula, the solution of the closed-loop system in (3) can be expressed as $X(t) = \Phi(t,0)X(0) + \int_0^t a(s)\Phi(t,s)(B\otimes I_n)\,\mathrm{d}W(s)$, leading to the convergence in mean square of $X(t)$ to a random vector by Cauchy's criterion [27].

By assumption (H_2), for $\forall\epsilon > 0$, there exists $T_0 > 0$ such that $\int_{T_0}^\infty a^2(s)\,\mathrm{d}s < \epsilon$. By (7) and (8), for the above $\epsilon > 0$, there exists $T_1 > T_0$ such that for $\forall t_2\geq t_1 > T_1$, $\max(\parallel\Phi(t_2,t_3) - \Phi(t_1,t_3)\parallel_1, \parallel\Phi(t_2,t_3) - \Phi(t_1,t_3)\parallel_\infty) < \epsilon$, $\forall t_3\in[0,T_0]$.

Let $X_2(t) = \int_0^t a(s)\Phi(t,s)(B\otimes I_n)\,\mathrm{d}W(s)$. Then

$$X_2(t_2) - X_2(t_1) = \int_0^{t_2} a(s)[\Phi(t_2,s) - \Phi(t_1,s)](B\otimes I_n)\,\mathrm{d}W(s) + \int_{t_1}^{t_2} a(s)\Phi(t_1,s)(B\otimes I_n)\,\mathrm{d}W(s)\triangleq X_{21} + X_{22}.$$

Therefore, $E\parallel X_2(t_2) - X_2(t_1)\parallel^2 \leq 2E\parallel X_{21}\parallel^2 + 2E\parallel X_{22}\parallel^2$. Obviously, $E\parallel X_{21}\parallel^2 = \int_0^{t_2} a^2(s)\parallel[\Phi(t_2,s) - \Phi(t_1,s)](B\otimes I_n)\parallel_F^2\,\mathrm{d}s \leq nN\parallel B\otimes I_n\parallel_1^2\times\epsilon^2\int_0^\infty a^2(s)\,\mathrm{d}s\triangleq M_1\epsilon^2$ and $\int_{T_0}^{t_2} a^2(s)\left\|[\Phi(t_2,s) - \Phi(t_1,s)](B\otimes I_n)\right\|_F^2\,\mathrm{d}s < 4nNM^2\parallel B\otimes I_n\parallel_1^2\epsilon\triangleq M_2\epsilon$, one obtains that $E\parallel X_{21}\parallel^2\leq M_1\epsilon^2 + M_2\epsilon$. Similarly, $E\parallel X_{22}\parallel^2\leq nNM^2\parallel B\otimes I_n\parallel_1^2\epsilon\triangleq(M_2/4)\epsilon$. Thus, $E\parallel X_2(t_2) - X_2(t_1)\parallel^2\leq 2M_1\epsilon^2 + (5/2)M_2\epsilon$.

Due to Cauchy's criterion and arbitrariness of ϵ, there exists X_2^* such that $X_2(t)$ converges to X_2^* in mean square. Therefore, $X(t)$ converges in mean square to X^*. By (7), one obtains $X^* = \left[T\mathrm{diag}(1,0,\ldots,0)T^{-1}\otimes I_n\right]X(0) + X_2^*$ □

3 Main results

3.1 Mean square bipartite consensus

Theorem 1: For the system in (1), the protocol in (2) is a mean square bipartite consensus protocol if and only if assumptions $(H_1)-(H_4)$ hold.

In what follows, we prove Theorem 1 step by step.

Proof: We prove the sufficiency in the three steps.

S.I. *Construct the consensus grouping:* By assumption (H_3), vertex set $\mathscr{V}$ of $\mathscr{G}$ can be divided into two subsets $\mathscr{V}_1$, $\mathscr{V}_2$, where $\mathscr{V}_1\cup\mathscr{V}_2 = \mathscr{V}$ and $\mathscr{V}_1\cap\mathscr{V}_2 = \phi$, and furthermore $a_{ij}\geq 0$ for $\forall i,j\in\mathscr{V}_k, k\in\{1,2\}$ and $a_{ij}\leq 0$ for $\forall i\in\mathscr{V}_k, j\in\mathscr{V}_l, k\neq l, k,l\in\{1,2\}$. Since vertices can be renumbered as needed, we may assume $\mathscr{V}_1 = \{1,\ldots,N_0\}$ and $\mathscr{V}_2 = \{N_0+1,\ldots,N\}$. Taking $\rho_i = 1$ for $i\in\mathscr{V}_1$ and $\rho_j = -1$ for $j\in\mathscr{V}_2$, by the definition of Laplacian $\mathscr{L}$, one has $\mathscr{L}\rho = \mathbf{0}$.

S.II. *Prove* $\lim_{t\to\infty} E\parallel X(t) - \rho\otimes c^*\parallel^2 = 0$*:* From Lemma 5 we know that $X(t)$ converges to X^* in mean square, i.e. $\lim_{t\to\infty} E\parallel X(t) - X^*\parallel^2 = 0$. Without loss of generality one may assume that $x_1(t)$ converges to c^* in mean square, i.e. $\lim_{t\to\infty} E\parallel x_1(t) - c^*\parallel^2 = 0$. Our goal is to show that $x_i(t)$ converges in mean square to $\rho_i c^*$, i.e. $\lim_{t\to\infty} E\parallel x_i(t) - \rho_i c^*\parallel^2 = 0$, $i = 2,\ldots,N$.

Let $\delta(t) = \left(\delta_2^{\mathrm{T}}(t),\ldots,\delta_N^{\mathrm{T}}(t)\right)^{\mathrm{T}}$, where $\delta_i(t)\triangleq x_i(t) - \rho_i x_1(t)$, $i = 2,\ldots,N$. It suffices to show that $\lim_{t\to\infty} E\parallel\delta(t)\parallel^2 = 0$. To this end, we analyse the evolution of the error dynamics, and use the similar non-singular matrix S as introduced in [14], i.e.

$$S\triangleq\begin{pmatrix}\frac{1}{N} & \frac{1}{N}\cdots\frac{1}{N} & -\frac{1}{N}\cdots-\frac{1}{N}\\ -\mathbf{1}_{N_0-1} & I_{N_0-1} & 0\\ -\mathbf{1}_{N-N_0} & 0 & -I_{N-N_0}\end{pmatrix} = \begin{pmatrix}\frac{1}{N}\rho^{\mathrm{T}}\\ S_2\end{pmatrix}.$$

Let $\tilde{\delta}(t)\triangleq(S\otimes I_n)X(t)$, and hence, $\tilde{\delta}(t) = (\varsigma^{\mathrm{T}}(t),\delta^{\mathrm{T}}(t))^{\mathrm{T}}$, where $\varsigma(t)\triangleq(1/N)(\rho^{\mathrm{T}}\otimes I_n)X(t)$. Since

$$S\mathscr{L}S^{-1} = \begin{pmatrix}0 & \gamma^{\mathrm{T}}\\ 0 & L_2\end{pmatrix},$$

by (3), one has

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$$\mathrm{d}\delta(t) = -a(t)(L_2 \otimes I_n)\delta(t)\,\mathrm{d}t + a(t)(S_2B \otimes I_n)\,\mathrm{d}W(s). \quad (11)$$

It is straightforward that L_2 and $\mathscr{L}$ have the same eigenvalues except one zero eigenvalue. Therefore, there exists an invertible matrix Q such that $Q^{-1}L_2Q = \mathrm{diag}(J_2, \ldots, J_s)$, where $J_2, \ldots, J_s$ are defined as in (6). Notice that the state transition matrix of (11) is $\Phi_2(t, t_0) = (Q \otimes I_n)\big[\mathrm{diag}\big(\Phi_{J_2^{\lambda_2}}(t, t_0), \ldots, \Phi_{J_s^{\lambda_s}}(t, t_0)\big)\big](Q^{-1} \otimes I_n)$, where $\Phi_{J_k^{\lambda_k}}(t, t_0)$, $k = 2, \ldots, s$ are defined as in Lemma 3. Therefore $\lim_{t\to\infty}\Phi_2(t, t_0) = \mathbf{0}$, i.e. for $\forall\epsilon > 0$, $\exists T_2 > T_1$ such that for any $t > T_2$, $\| \Phi_2(t, t_0) \| < \epsilon$, and hence $\exists M_{L_2} > 0$ such that $\max\{ \| \Phi_2(t, t_0) \|_1, \| \Phi_2(t, t_0) \|_\infty \} \le M_{L_2} < \infty, \forall t \ge t_0 \ge 0$.

By *Itô* formula,

$$E \| \delta(t) \|^2 = \Big\| \Phi_2(t, 0)\delta(0) \Big\|^2 + \int_0^t a^2(s) \| \Phi_2(t, s)(S_2B \otimes I_n) \|_F^2 \,\mathrm{d}s.$$

Due to the fact that $\int_{T_0}^{\infty} a^2(s)\,\mathrm{d}s < \epsilon$, one has

$$\int_{T_0}^{t} a^2(s) \| \Phi_2(t, s)(S_2B \otimes I_n) \|_F^2 \,\mathrm{d}s \le nNM_{L_2}^2 \| S_2B \otimes I_n \|_1^2 \,\epsilon \triangleq M_3\epsilon.$$

This, together with $\int_0^{T_0} a^2(s) \| \Phi_2(t, s)(S_2B \otimes I_n) \|_F^2 \,\mathrm{d}s = \| \Phi_2(t, \xi)(S_2B \otimes I_n) \|_F^2 \times \int_0^{T_0} a^2(s)\,\mathrm{d}s \le nN \| \Phi_2(t, \xi) \|_1^2 \| (S_2B \otimes I_n) \|_1^2 \int_0^{\infty} a^2(s)\,\mathrm{d}s$, $\xi \in (0, T_0)$, yields that for any $t > T_2$, $E \| \delta(t) \|^2 \le \| \delta(0) \|^2 \epsilon^2 + nN \| (S_2B \otimes I_n) \|_1^2 \int_0^{\infty} a^2(s)\,\mathrm{d}s\,\epsilon^2 + M_3\epsilon$. Since ϵ is arbitrary, $\lim_{t\to\infty} E \| \delta(t) \|^2 = 0$. Thus $\lim_{t\to\infty} E \| X(t) - \rho \otimes c^* \|^2 = 0$.

S.III. *Determine the statistic characteristics of c^*:* It is evident that $\rho \otimes c^* = X^* = \big[T\mathrm{diag}(1, 0, \ldots, 0)T^{-1} \otimes I_n\big]X(0) + X_2^*$. Then $\rho \otimes Ec^* = EX^* = \big[T\mathrm{diag}(1, 0, \ldots, 0)T^{-1} \otimes I_n\big]\,X(0)$.

Suppose the first column of T is w_r and the first row of T^{-1} is w_l^{T}. Then $\rho \otimes Ec^* = (w_rw_l^{\mathrm{T}} \otimes I_n)X(0)$. By (6), one has $\mathscr{L}T = TJ$ and $T^{-1}\mathscr{L} = JT^{-1}$. Thus $\mathscr{L}w_r = \mathbf{0}$ and $w_l^{\mathrm{T}}\mathscr{L} = \mathbf{0}$, i.e. w_r and w_l are the right and left eigenvectors of $\mathscr{L}$ corresponding to eigenvalue 0, respectively. Since $\mathscr{L}$ has only one zero eigenvalue, the eigenspace corresponding to the zero eigenvalue is 1-dimensional (1D). From S.I we know that $\mathscr{L}\rho = 0$. Therefore, $w_r = \kappa\rho$ ($\kappa \in \mathbb{R}, \kappa \ne 0$) and $\rho \otimes Ec^* = \rho \otimes \big[\kappa(w_l^{\mathrm{T}} \otimes I_n)X(0)\big]$. Immediately, one obtains $Ec^* = \big[\kappa(w_l^{\mathrm{T}} \otimes I_n)X(0)\big]$. Clearly, w_l is dependent on the communication topology and hence Ec^* is determined by the initial state $X(0)$ and the communication topology among agents.

By (7), it is obvious that $\Phi(,)$ is uniformly bounded. Then, for any given $\epsilon > 0$, there exists $T_3 > T_2$ such that

$$\int_{T_3}^{\infty} a^2(s) \| \Phi(t, s)(B \otimes I_n)(B \otimes I_n)^{\mathrm{T}}\Phi^{\mathrm{T}}(t, s) \| \,\mathrm{d}s < \epsilon,$$
$$\int_{T_3}^{\infty} a^2(s) \| (w_rw_l^{\mathrm{T}} \otimes I_n)(B \otimes I_n)(B \otimes I_n)^{\mathrm{T}}(w_rw_l^{\mathrm{T}} \otimes I_n)^{\mathrm{T}} \| \,\mathrm{d}s < \epsilon. \quad (12)$$

Let $X_3(t) = \int_0^{T_3} a^2(s)\Phi(t, s)(B \otimes I_n)(B \otimes I_n)^{\mathrm{T}}\Phi^{\mathrm{T}}(t, s)\,\mathrm{d}s$. Then there exists T_4 such that for all $t > T_4$, $\| X_3(t) - \int_0^{T_3} a^2(s)(w_rw_l^{\mathrm{T}} \otimes I_n)(B \otimes I_n)(B \otimes I_n)^{\mathrm{T}}(w_rw_l^{\mathrm{T}} \otimes I_n)^{\mathrm{T}}\,\mathrm{d}s \| < \epsilon$. Combining this with (12),

$$\lim_{t\to\infty} \int_0^t a^2(s)\Phi(t, s)(B \otimes I_n)(B \otimes I_n)^{\mathrm{T}}\Phi^{\mathrm{T}}(t, s)\,\mathrm{d}s = \int_0^{\infty} a^2(s)(w_rw_l^{\mathrm{T}}BB^{\mathrm{T}}w_lw_r^{\mathrm{T}} \otimes I_n)\,\mathrm{d}s = \Omega(\rho\rho^{\mathrm{T}} \otimes I_n),$$

where $\Omega = \kappa^2(w_l^{\mathrm{T}}BB^{\mathrm{T}}w_l)\int_0^{\infty} a^2(s)\,\mathrm{d}s$. According to this and $D(X^*) = \lim_{t\to\infty}\int_0^t a(s)\Phi(t, s)(B \otimes I_n)(B \otimes I_n)^{\mathrm{T}}\Phi^{\mathrm{T}}(t, s)\,\mathrm{d}s$, it can be obtained that $D(c^*) = \Omega I_n$. Therefore, $E \| c^* \|^2 < \infty$. By Definition 1, protocol (2) is a mean square bipartite consensus protocol.

Necessity: We prove the necessity in the four steps, as follows.

N.I. *Prove the necessity of* (H_1), i.e. $\int_0^{\infty} a(s)\,\mathrm{d}s = +\infty$: If (H_1) does not hold, then there exists a constant $\bar{a} > 0$ such that $\int_0^{\infty} a(s)\,\mathrm{d}s = \bar{a}$. Therefore, $\lim_{t\to\infty} e^{-\int_0^t a(s)\,\mathrm{d}s\mathscr{L}} = e^{-\bar{a}\mathscr{L}}$. Since $e^{-\bar{a}\mathscr{L}}$ is invertible, we obtain that $\mathrm{rank}(\lim_{t\to\infty}\Phi(t, 0)) = \mathrm{rank}\big(e^{-\bar{a}\mathscr{L}} \otimes I_n\big) = nN$. However, from Lemma 4 we know that $\lim_{t\to\infty}\Phi(t, 0) = \rho v \otimes I_n$, and hence $\mathrm{rank}(\lim_{t\to\infty}\Phi(t, 0)) \le n$. This is a contradiction. Thus, (H_1) holds.

N.II. *Prove that Laplacian $\mathscr{L}$ has and only has one zero eigenvalue*: First, we prove that 0 is an eigenvalue of $\mathscr{L}$. If 0 is not an eigenvalue of $\mathscr{L}$, then $-\mathscr{L}$ is a *Hurwitz* matrix. From (H_1) and Lemma 3 we know that $\lim_{t\to\infty}\Phi(t, 0) = \mathbf{0}$. This together with Lemma 4 yields that $\rho \otimes Ec^* = EY^*$. Obviously, EY^* has nothing to do with the initial state $X(0)$, and thus Ec^* is independent from $X(0)$. This contradicts the statement that Ec^* is determined by $X(0)$ and the communication topology among agents. Therefore, 0 is an eigenvalue of $\mathscr{L}$.

Suppose J_1^0 is a Jordan block corresponding to eigenvalue 0. Then it is 1D. Suppose by contradiction that J_1^0 is N_1-dimensional and $N_1 > 1$.

By (5) and assumption (H_1), we get that $\lim_{t\to\infty} e^{-\int_0^t a(s)\,\mathrm{d}s J_1^0}$ does not exist and hence $\lim_{t\to\infty}\Phi(t, 0)$ does not exist. This contradicts Lemma 4. Thus J_1^0 is 1D.

Second, we prove that 0 is an eigenvalue of algebraic multiplicity 1. Assume the multiplicity of 0 is m. If $m > 1$, then we take $m = 2$ as an example. In this case, due to the fact that every Jordan block corresponding to eigenvalue 0 is 1D, one has $T^{-1}\mathscr{L}T = \mathrm{diag}(0, 0, J_3, \ldots, J_s)$. Thus

$$\Phi_\infty \otimes I_n \triangleq \lim_{t\to\infty}\Phi(t, 0) = \lim_{t\to\infty}\Big[e^{-\int_0^t a(s)\,\mathrm{d}s\mathscr{L}} \otimes I_n\Big] = T\mathrm{diag}(1, 1, 0, \ldots, 0)T^{-1} \otimes I_n$$

which implies that $\mathrm{rank}(\Phi_\infty) = 2$. However, from Lemma 4 we know that $\mathrm{rank}(\Phi_\infty) \le 1$. This is a contradiction. Other cases can be proved similarly. Therefore, $\mathscr{L}$ has exactly one zero eigenvalue.

N.III. *Prove the necessity of assumptions* (H_3) *and* (H_4)*:* From the above analysis, we know that $\mathscr{L}$ has exactly one zero eigenvalue. Combining this with (H_1) obtains (7). This together with Lemma 4 gives

$$T\mathrm{diag}(1, 0, \ldots, 0)T^{-1} = \rho v^{\mathrm{T}}. \quad (13)$$

Note that w_r is the first column of T. Then by (6), $\mathscr{L}w_r = 0$. From (13), it follows that $w_r = \rho k^*$, where $k^* = v^Tw_r \in \mathbb{R}$. Thus, $\mathscr{L}\rho = 0$. By the definition of *Laplacian* $\mathscr{L}$, we obtain that for any j, $\rho_j\sum_{k\ne j}|a_{jk}| = \sum_{k\ne j}\rho_k a_{jk}$, $j = 1, \ldots, N$. Noticing that $\rho_j = \pm 1$ and $\rho_j^2 = 1$, $j = 1, \ldots, N$, we get $\sum_{k\ne j}|a_{jk}| = \sum_{k\ne j}\rho_j\rho_k a_{jk}$. Thus, $\rho_j\rho_k a_{jk} = |a_{jk}| \ge 0$. This implies that $a_{jk} = 0$ or all non-zero terms $\rho_k a_{jk}$ have the same sign with ρ_j. Let $V_1 = \{j | \rho_j = 1\}$ and $V_2 = \{j | \rho_j = -1\}$. Then $V_1 \cup V_2 = V$ and $V_1 \bigcap V_2 = \varnothing$. Obviously,

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if $j \in V_q (q \in \{1,2\})$, then by $\rho_j \rho_k a_{jk} = |a_{jk}| \geq 0$, we have $a_{jk} \geq 0, k \in V_q$ and $a_{jk} \leq 0, k \in V_r, r \neq q, r \in \{1,2\}$. By definition, $\mathcal{G}$ is structurally balanced, or equivalently, (H_3) holds.

From *N.II.*, (H_3) and Lemma 1, we know that $\mathcal{G}$ has a spanning tree, i.e. (H_4) holds.

N.IV. *Prove the necessity of assumption* (H_2)*:* By contradiction, we assume $\int_0^\infty a^2(s)\,\mathrm{d}s = \infty$. Note that w_l^{T} is the first row of T^{-1} and $w_l^{\mathrm{T}}\mathcal{L} = 0$. Then by (3), one has $\mathrm{d}\big((w_l^{\mathrm{T}} \otimes I_n)X(t)\big) = a(t)(w_l^{\mathrm{T}} \otimes I_n)(B \otimes I_n)\,\mathrm{d}W(t)$, i.e., $(w_l^{\mathrm{T}} \otimes I_n)X(t) = (w_l^{\mathrm{T}} \otimes I_n)X(0) + (w_l^{\mathrm{T}} \otimes I_n)(B \otimes I_n)\int_0^t a(s)\,\mathrm{d}W(s)$. By Definition 1, $X(t)$ converges in mean square to $\rho \otimes c^*$ with $E \parallel c^* \parallel^2 < \infty$. Then, as $t \to \infty$, $(w_l^{\mathrm{T}} \otimes I_n)(B \otimes I_n)\int_0^t a(s)\,\mathrm{d}W(s)$ converges in mean square to a random vector X_ω with $E \parallel X_\omega \parallel^2 < \infty$. Thus, $\lim_{t \to \infty} E \parallel (w_l^{\mathrm{T}} \otimes I_n)(B \otimes I_n)\int_0^t a(s)\,\mathrm{d}W(s) \parallel^2 = E \parallel X_\omega \parallel^2 < \infty$. On the other hand,

$$\lim_{t \to \infty} E \parallel (w_l^{\mathrm{T}} \otimes I_n)(B \otimes I_n)\int_0^t a(s) \times \mathrm{d}W(s) \parallel^2$$
$$= \lim_{t \to \infty} \mathrm{tr}(w_l^{\mathrm{T}} B B^{\mathrm{T}} w_l \otimes I_n)\int_0^t a^2(s)\,\mathrm{d}s = \infty.$$

This leads to a contradiction. Thus, (H_2) holds.

We thus complete the proof.

□

On the premise of structural balance, a bipartite consensus problem can be converted into a conventional consensus problem on a standard graph with the help of a gauge transformation [8, 26, 28]. This is also true in the case with noise [25]. Hence, if $\mathcal{G}$ is weight balanced, then under assumptions $(H_1) - (H_4)$ bipartite average consensus can be derived from average consensus results in [19].

Corollary 1: If $\mathcal{G}$ is weight balanced and assumptions $(H_1) - (H_4)$ hold, then $\lim_{t \to \infty} E \parallel X(t) - \mathscr{D}\mathbf{1}_N \otimes c^* \parallel^2 = 0$, where

$$\mathscr{D} = \mathrm{diag}(d_1, \ldots, d_N)(d_i \in \{\pm 1\}),$$
$$Ec^* = \frac{1}{N}\big(\mathbf{1}_N^T \mathscr{D} \otimes I_n\big)X(0), \quad \text{and}$$

$$Dc^* = \frac{(\sum_{i=1}^N \sum_{j=1}^N a_{ij}^2)\int_0^\infty a^2(s)\,\mathrm{d}s}{N^2} I_n.$$

Remark 4: The key tool in [19] is symmetrised graph, which heavily relies on the balance of $\mathcal{G}$. However, in Theorem 1 we only require that $\mathcal{G}$ has a spanning tree, but not on its weight balance. This means that the analysis tool in [19] does not apply directly to the sufficiency proof of Theorem 1.

In the necessity proof of Theorem 1, structural balance does not exist as a prerequist but as a condition to be proved. Therefore, gauge transformation is invalid and thus the results in [19–22] are not directly applicable to Theorem 1. Moreover, convergence analysis of [19] heavily relies on the tool of balanced graph. So, the results in [19] cannot be applied here neither.

From Theorem 1 one sees that structural balance is necessary for ensuring the mean square bipartite consensus in the presence of noise. One may be further interested to know the evolution of the closed-loop system in (3) for structurally unbalanced $\mathcal{G}$. For that purpose, we give the following assumptions.

(H_3') $\mathcal{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ is structurally unbalanced.
(H_4') $\mathcal{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ does not contain an ISB subgraph.

An input-isolated subgraph of $\mathcal{G}$ means that agents of the subgraph cannot receive information from other agents of $\mathcal{G}$. If such a subgraph is structurally balanced, then its agents may achieve bipartite consensus independent from the remaining agents of $\mathcal{G}$.

Theorem 2: The protocol in (2) is mean square stabilising under assumptions (H_1), (H_2), (H_3') and (H_4'), i.e. for any given $X(0)$, $\lim_{t \to \infty} E \parallel X(t) \parallel^2 = 0$.

On the other hand, assumptions (H_1), (H_3') and (H_4') must hold if the protocol in (2) is mean square stabilising.

Proof: Since (H_3') and (H_4') hold, Lemma 2 implies that all the eigenvalues of $\mathscr{L}$ are in the open right half plane. For the closed-loop system in (3), adopting arguments similar to the proof after (11) in the sufficiency proof of Theorem 1, we know from (H_1) and (H_2) that for any given $X(0)$, $\lim_{t \to \infty} E \parallel X(t) \parallel^2 = 0$, i.e. (2) is mean square stabilising.

On the other hand, if protocol (2) is mean square stabilising, then repeating the similar procedure as in *N.I* of Theorem 1, we obtain that (H_1) holds. According to *N.II* and Lemma 4, we obtain that 0 is not an eigenvalue of $\mathscr{L}$. Therefore, all the eigenvalues of $\mathscr{L}$ are in the open right half-plane. Combining Lemma 2, assumptions (H_3') and (H_4') hold. □

Theorem 2 indicates that, regardless of the initial state, the agent state under protocol (2) converges to zero in mean square under mild conditions, for structurally unbalanced $\mathcal{G}$.

3.2 Bipartite consensus in the absence of measurement noise

In this section, we show that under (H_1), (H_3) *and* (H_4), the protocol in (2) is also a bipartite consensus protocol in the absence of measurement noise.

Suppose $\varphi_{ji}(t) \equiv \mathbf{0}$, the protocol in (2) is reduced to

$$u_i(t) = a(t)\left[\sum_{j \in \mathscr{N}_i} |a_{ij}|\big(\mathrm{sgn}(a_{ij})x_j(t) - x_i(t)\big)\right], \tag{14}$$

Notice that for $a(t) \equiv 1$, protocol (14) is equivalent to the protocol in [8, 11–13, 26].

Applying (14) to (1) leads to the closed-loop system $\dot{X}(t) = -a(t)(\mathscr{L} \otimes I_n)X(t)$.

Theorem 3: For the system in (1), the protocol in (14) is a bipartite consensus protocol if and only if (H_1), (H_3) and (H_4) hold.

Proof: Using the similar arguments as in Theorem 1, one immediately obtains the above theorem. □

Remark 5: The role of (H_2) can be seen from Theorems 1 and 3. Indeed, (H_1) is the only requirement for $a(t)$ to ensure a bipartite consensus in the case without measurement noise. However, (H_1) itself is not sufficient in the case with noise, since in such a case the stochastic process $X(t)$ is required to be square integrable, which can only be true in the presence of (H_2).

Remark 6: If we take $a(t) \equiv 1$, then protocol (14) becomes the protocol in [8, 11–13, 26]. In this situation, assumption (H_1) naturally holds, and therefore Theorem 3 degenerates to Theorem 1 in [12], i.e. bipartite consensus is achieved if and only if $\mathcal{G}$ is structurally balanced and has a spanning tree. Compared with [8], where $\mathcal{G}$ is required to be strongly connected, here having a spanning tree is less stringent.

Corollary 2: For the system in (1), the protocol in (14) is stabilizing if and only if assumptions (H_1), (H_3') and (H_4') hold.

Remark 7: When $a(t) \equiv 1$, Corollary 2 is consistent with Theorem 1 in [12]. In [29], for the system to be stabilizing, $\mathcal{G}$ is required to be strongly connected. Clearly, it is a special case of (H_4').

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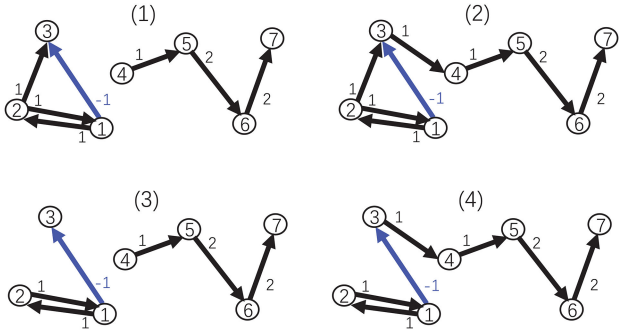

Fig. 1 *Four types of communications of the signed digraph*

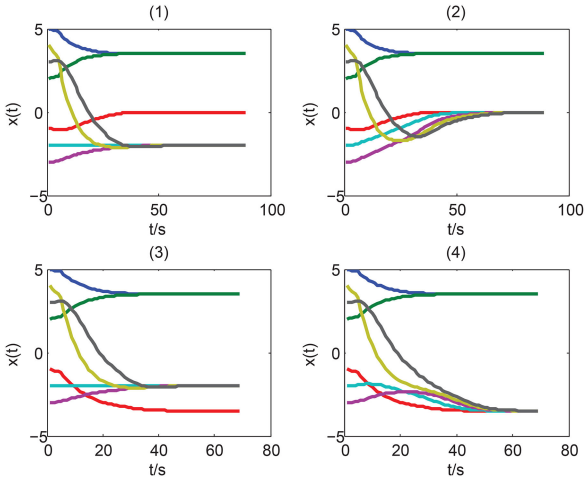

Fig. 2 *State trajectories of seven agents with* $a(t) = 1$

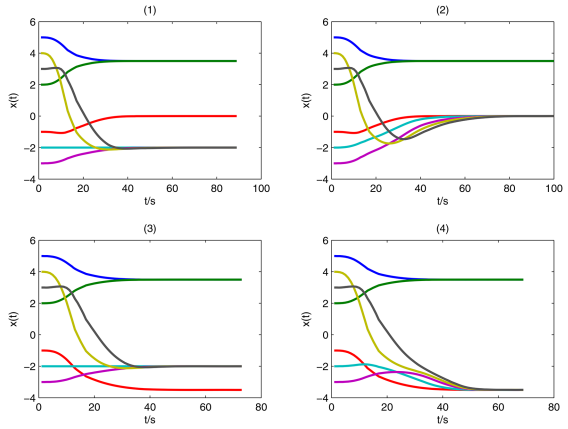

Fig. 3 *State trajectories of seven agents with* $a(t) = t^2$

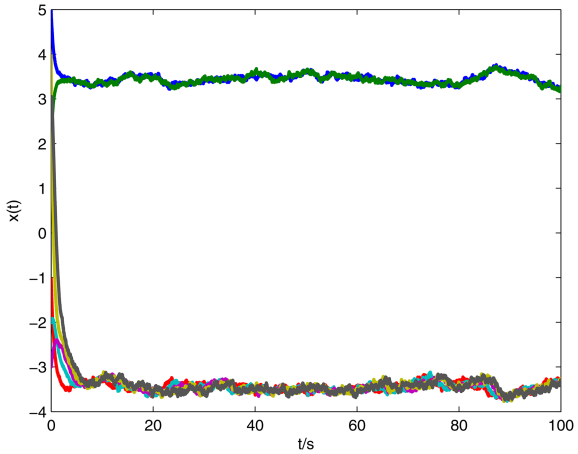

Fig. 4 *Fluctuations of seven agents with* $a(t) = 1$

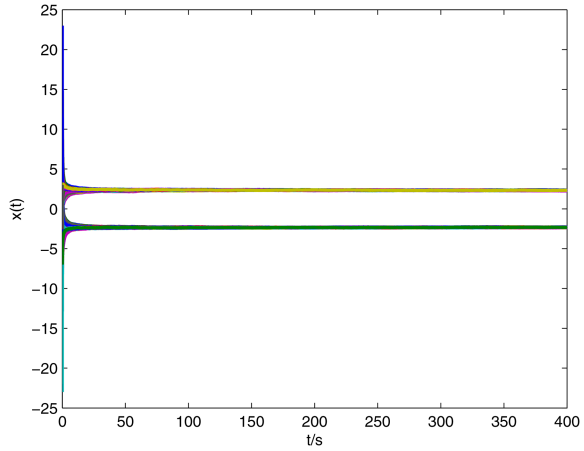

Fig. 5 *Bipartite consensus of 100 agents in the case with noise*

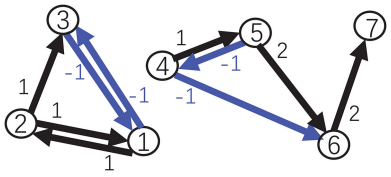

Fig. 6 *Signed digraph with structural unbalance and no ISB subgraph*

4 Numerical example

Consider a signed digraph $\mathcal{G}$ with $N = 7$, as shown in Fig. 1, where four types of communications are considered and blue lines represent competitive interactions among agents.

In (1), (2) and (3) of Fig. 1, $\mathcal{G}$ either does not have a spanning tree or is structurally unbalanced, while in (4), $\mathcal{G}$ has a spanning tree and is also structurally balanced. If we take $a(t) = 1$ and $a(t) = t^2$, respectively, then seven state trajectories can be found in Figs. 2 and 3.

One can observe that only when $\mathcal{G}$ satisfies (4) in Fig. 1, can bipartite consensus be achieved. This is consistent with Theorem 3. Comparing with (4) in Figs. 2 and 3, one can see that the time of reaching bipartite consensus is about 60 s and 50 s respectively. This means that the faster $\int_0^t a(s)\,ds$ approaches ∞, the faster the considered system reaches bipartite consensus.

Let $\mathcal{G}$ satisfy (4) in Fig. 1. In the presence of measurement noise, if we still take $a(t) = 1$, then the states of the seven agents fluctuate randomly with time, as shown in Fig. 4.

Clearly, as time goes on, the fluctuation will not die off. We know from Theorem 1 that (H_1) and (H_2) are necessary and sufficient conditions on $a(t)$ for achieving bipartite consensus in the presence of noise. Hence let $a(t) = 1/(t+1)$. To demonstrate the convergence performance more clearly, a signed digraph $\mathcal{G}$ of 100 nodes is considered. The stochastic protocol in (2) is used in Fig. 5, which shows the bipartite consensus in the presence of measurement noise.

In particular, if the signed digraph $\mathcal{G}$ with $N = 7$ is structurally unbalanced and does not contain an ISB subgraph, such as in Fig. 6, then we apply the protocol in (2) to the system in (1) with $a(t) = 1/(t+1)$ satisfying (H_1) and (H_2).

In this case, the conditions of Theorem 2 hold. The initial state is given by $X(0) = (10, -21, -1, 7, -3, 6, -2)^{\mathrm{T}}$. The states of the closed-loop system are described in Fig. 7.

It can be seen that mean square stabilising is achieved asymptotically.

5 Concluding remarks

Bipartite consensus for first-order integrator MASs is investigated in the context of measurement noise, where existing protocols often lead to fluctuations. A new stochastic type protocol with time-varying consensus gain is proposed, and necessary and sufficient conditions for ensuring mean square bipartite consensus

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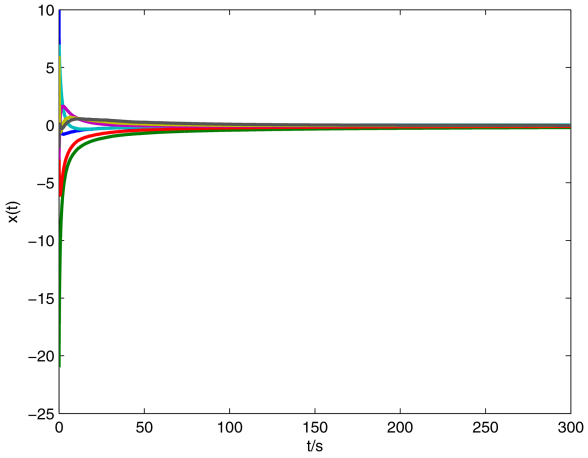

Fig. 7 *Seven state trajectories under signed digraph with structural unbalance and no ISB subgraph*

are given. Moreover, under the assumption of structural unbalance, the states of the agents are proved to converge to zero in mean square, regardless of the initial states. The results in this work are only for first-order integrator MASs under fixed topologies, and hence time-varying topology becomes interesting in our future work.

6 Acknowledgments

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